Q.1. Find the linear approximation of $f(x) = x^2$ at $x = 5$ & use it to approximate $(5.27)^2$

Sol^n:- The Linear approximation of $f(x)$ at $x = a$ is given by $L(x) = f(a) + f'(a)(x - a)$.

Here, $a = 5$ & $f(x) = x^2$.

Then $f'(x) = 2x \Rightarrow f'(5) = 2(5) = 10$

So, $L(x) = f(5) + f'(5)(x-5)$

$$= 25 + 10(x-5)$$

$$= 10x - 25$$

Then the approximation of $(5.27)^2$ is,

$$L(5.27) = 10(5.27) - 25$$

$$= 52.7 - 25$$

$$= 27.7$$

Q.2 If $g(x) = \int_{x^2}^{5} \sin(3t)\,dt$, then find $g'(x)$.

Using the General version of FTC we get,

$$\frac{d}{dx}\int_{k(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(k(x))\cdot k'(x).$$

Here, $h(x) = 5 \Rightarrow h'(x) = 0$

$k(x) = x^2 \Rightarrow k'(x) = 2x$

& $f(h(x)) = \sin(3(h(x)))$

$= \sin(3(5))$

$= \sin(15)$

$f(k(x)) = \sin(3(k(x)))$

$= \sin(3(x^2))$

$= \sin(3x^2)$

Therefore,

$$g'(x) = \frac{d}{dx}\int_{x^2}^{5} \sin(3t)\,dt$$

$$= \sin(15)\cdot 0 - \sin(3x^2)\cdot 2x$$

$$= -2x\sin(3x^2)$$

Q.3. If $f(x) = \frac{x+7}{5x+2}$, then find $f^{-1}(1)$.

Sol^n:- First identify the domain & range of f.

$$\text{Domain}(f) = \{x \mid 5x+2 \neq 0\}$$
$$= \{x \text{ is real} \mid x \neq -\tfrac{2}{5}\}$$

$$\text{Range}(f) = \{x \mid x = \frac{y+7}{5y+2}\}$$
$$= \{x \mid 5xy + 2x = y+7\}$$
$$= \{x \mid 5xy - y = -2x+7\}$$
$$= \{x \mid y(5x-1) = -2x+7\}$$
$$= \{x \mid \boxed{y = \frac{-2x+7}{5x-1}}\}$$
$$= \{x \mid 5x-1 \neq 0\}$$
$$= \{x \mid x \neq \tfrac{1}{5}\}$$

Now in this process we get

$$f^{-1}(x) = \frac{-2x+7}{5x-1}$$

So, $f^{-1}(1) = \frac{-2(1)+7}{5(1)-1} = \frac{-2+7}{5-1} = \frac{5}{4}$

Q.4. Find the number of vertical & horizontal asymptotes of the function $f(x) = \frac{3}{x-4}$

As $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \frac{3}{x-4} = -\infty$

& $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{3}{x-4} = +\infty$

$\Big] \rightsquigarrow x = 4$ is a vertical asymptote.

Also, $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{3}{x-4} = 0 \; \Big] \to y = 0$ is a horizontal asymptote

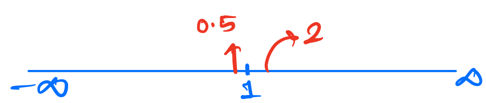
So, one horizontal & one vertical asymptote.

Q.5. Find the inflection point for $f(x) = x^3 - 3x^2 + 4x$, if exists.

$f(x) = x^3 - 3x^2 + 4x$

$f'(x) = 3x^2 - 6x + 4$

$f''(x) = 6x - 6$



Solve for $f''(x) = 0 \Rightarrow 6x - 6 = 0 \Rightarrow x = 1$.

Now, $f''(0.5) = 6(0.5) - 6 = 3 - 6 = -3 < 0$

$f''(2) = 6(2) - 6 = 12 - 6 = 6 > 0$

$\Big\}$ $f''(x)$ changes sign at $x=1$

$x = 1$ is an inflection pt

Q.6. If $f(x) = A\sin x + B\cos x + 7$ & $f(0) = 4$, $f'(\pi) = -2$, then find A & B.

$f(x) = A\sin x + B\cos x + 7$

So, $f(0) = A\sin 0 + B\cos 0 + 7$

$\Rightarrow 4 = A(0) + B(1) + 7$

$\Rightarrow 4 = B + 7 \Rightarrow B = -3$

Now, $f'(x) = A\cos x - B\sin x$

& $f'(\pi) = A\cos\pi - B\sin\pi$

$\Rightarrow -2 = A(-1) - B(0) = -A$

$\Rightarrow A = 2$

Q.7. Determine the area of the region $x = -y^2 + 10$ & $x = (y-2)^2$

For intersection pts, we have,

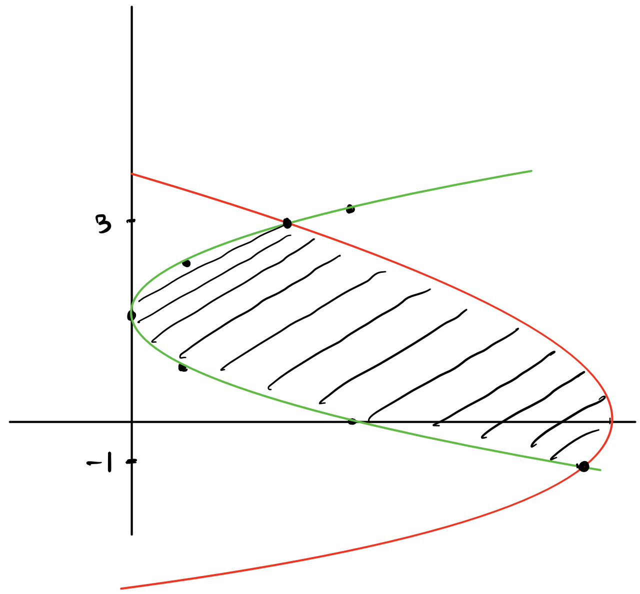
$-y^2 + 10 = (y-2)^2$

$\Rightarrow -y^2 + 10 = y^2 - 4y + 4$

$\Rightarrow 2y^2 - 4y - 6 = 0$

$\Rightarrow y^2 - 2y - 3 = 0 \Rightarrow (y-3)(y+1) = 0 \Rightarrow y = -1, y = 3$

Under standing the graph.



$$\text{Area} = \int_{-1}^{3} \left[(-y^2+10) - (y-2)^2\right] dy$$

$$= \int_{-1}^{3} \left[-y^2+10 - (y^2-4y+4)\right] dy$$

$$= \int_{-1}^{3} \left[-y^2+10-y^2+4y-4\right] dy$$

$$= \int_{-1}^{3} \left[-2y^2+4y+6\right] dy$$

$$= \left[-2\left(\frac{y^3}{3}\right) + 4\left(\frac{y^2}{2}\right) + 6(y)\right]_{-1}^{3}$$

$$= \left[-2\left(\frac{3^3}{3}\right) + 4\left(\frac{3^2}{2}\right) + 6(3)\right] - \left[-2\left(\frac{(-1)^3}{3}\right) + 4\left(\frac{(-1)^2}{2}\right) + 6(-1)\right]$$

$$= \left[-\cancel{18} + \cancel{18} + 18\right] - \left[\frac{2}{3} + 2 - 6\right]$$

$$= [18] - \left[\frac{2}{3} - 4\right]$$

$$= 18 - \frac{2}{3} + 4$$

$$= 22 - \frac{2}{3}$$

$$= \frac{66}{3} - \frac{2}{3}$$

$$= \frac{64}{3}$$

Q.8. For a constant "a", solve the following

$$5 + e^{7x} = 8e^{2a}$$

Given $\quad 5 + e^{7x} = 8e^{2a}$

Constants

So, $\quad e^{7x} = 8e^{2a} - 5$

$$\Rightarrow \ln(e^{7x}) = \ln(8e^{2a} - 5)$$

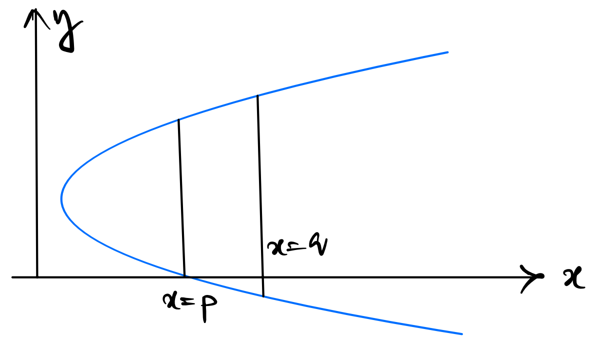
$$\Rightarrow 7x = \ln(8e^{2a} - 5)$$

$$\Rightarrow x = \frac{1}{7} \ln(8e^{2a} - 5)$$

Q.9. Find the values of a & b for which the graph of $x = 4y^2 - y + 3$ has a vertical secant line joining $(*, a)$ & $(**, b)$.

(i) $a = 0,\ b = \frac{1}{4}$; (ii) $a = 1,\ b = \frac{1}{4}$

Given $x = 4y^2 - y + 3$ is an equation of parabola.



So we need to check for which pair of a & b

* & ** are same.

(i) $a = 0, \quad b = \frac{1}{4}$ ✓

$* = 4a^2 - a + 3$
$= 4 \cdot 0^2 - 0 + 3$
$= 3$

$** = 4b^2 - b + 3$
$= 4\left(\frac{1}{4}\right)^2 - \frac{1}{4} + 3$
$= 4 \cdot \frac{1}{4^2} - \frac{1}{4} + 3$
$= \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} + 3$
$= 3$

(ii) $a = 1, \quad b = \frac{1}{4}$ ✗

$* = 4a^2 - a + 3$
$= 4(1)^2 - 1 + 3$
$= 4 - 1 + 3$
$= 6$

$** = 4b^2 - b + 3$
$= 4\left(\frac{1}{4}\right)^2 - \frac{1}{4} + 3$
$= 4 \cdot \frac{1}{4^2} - \frac{1}{4} + 3$
$= \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} + 3$
$= 3$

Q.10. Find $\lim_{x \to 1} \frac{\ln x}{2x-2}$.

$$\lim_{x \to 1} \frac{\ln x}{2x-2} \quad \left[\frac{0}{0}\right]$$

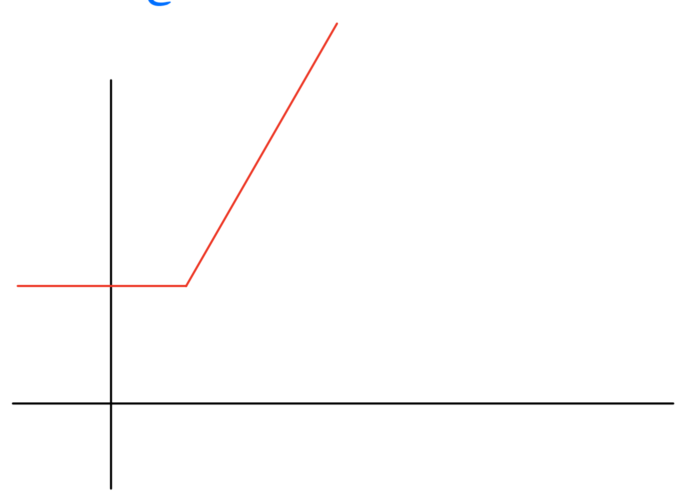
$$= \lim_{x \to 1} \frac{1/x}{2} \quad \left[\text{Not indeterminate}\right]$$

$$= \lim_{x \to 1} \frac{1}{2x} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$

Q.11 Analyze $f(x) = 2|5-x| + 2x$

$$f(x) = \begin{cases} 2(5-x) + 2x, & x \leq 5 \\ 2(x-5) + 2x, & x \geq 5 \end{cases}$$

$$= \begin{cases} 10, & x \leqslant 5 \\ 4x - 10, & x \geqslant 5 \end{cases}$$



$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} 10 = 10$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} (4x-10) = 4(5)-10 = 10$$

So $\lim_{x\to 5} f(x)$ exists. & $= 10$

$$f(5) = 10$$

So, $f(x)$ is continuous at $x=5$ & consequently $f(x)$ is continuous on $\mathbb{R}$.

$$\lim_{x\to 5^-} \frac{f(x)-f(5)}{x-5} = \lim_{x\to 5^-} \frac{10-10}{x-5} = \lim_{x\to 5^-} \frac{0}{x-5} = 0$$

$$\lim_{x\to 5^+} \frac{f(x)-f(5)}{x-5} = \lim_{x\to 5^+} \frac{(4x-10)-10}{x-5}$$

$$= \lim_{x\to 5^+} \frac{4x-20}{x-5}$$

$$= \lim_{x\to 5^+} \frac{4\cancel{(x-5)}}{\cancel{(x-5)}} = \lim_{x\to 5^+} 4 = 4$$

Hence, $\lim_{x\to 5^-} \frac{f(x)-f(5)}{x-5} \neq \lim_{x\to 5^+} \frac{f(x)-f(5)}{x-5}$

$\Rightarrow$ $f(x)$ is not differentiable at $x=5$.